Faculty of Science, Technology, Engineering and Mathematics
M337 Complex analysis

M337 Solutions to Specimen exam 3

There are alternative solutions to many of these questions. Any correct solution that is set out clearly is worth full marks.

Question 1

Let $w = -2 + 2i$.

(a) (i) $\dfrac{1}{w} = \dfrac{\overline{w}}{|w|^2} = \dfrac{-2-2i}{8} = -\dfrac{1+i}{4}$ 2

(ii) $\operatorname{Log} w = \log|w| + i \operatorname{Arg} w = \log\sqrt{8} + i\dfrac{3\pi}{4}$ 2

(iii) Since $|w| = \sqrt{8}$ and $\operatorname{Arg} w = 3\pi/4$, we have $w = \sqrt{8}e^{3i\pi/4}$. Hence

$$w^3 = (\sqrt{8}e^{3i\pi/4})^3 = 8\sqrt{8}e^{9i\pi/4} = 16\sqrt{2}e^{i\pi/4}.$$

Therefore

$$\operatorname{Log}(w^3) = \log|w^3| + i\operatorname{Arg}(w^3) = \log(16\sqrt{2}) + i\frac{\pi}{4}.$$ 3

(b) We have $w = \sqrt{8}e^{3i\pi/4}$. By HB A1 3.2, p17, noting that $8^{1/6} = \sqrt{2}$, the cube roots of w are

$$z_k = \sqrt{2}e^{i(\pi/4+2\pi k/3)}, \quad \text{for } k = 0, 1, 2.$$

That is,

$$z_0 = \sqrt{2}e^{i\pi/4}, \quad z_1 = \sqrt{2}e^{11i\pi/12}, \quad z_2 = \sqrt{2}e^{19i\pi/12}.$$ 3

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Question 2

(a) The function

$$f(z) = \frac{\sinh z}{z}$$

is analytic on $\mathbb{C} - \{0\}$ and has a singularity at 0. Observe that

$$\lim_{z\to 0} z f(z) = \lim_{z\to 0} \sinh z = \sinh 0 = 0.$$

Hence f has a removable singularity at 0, by HB B4 3.1, p58. 3

(b) The function

$$f(z) = \frac{\sin z}{(z-\pi)^3}$$

is analytic on $\mathbb{C} - \{\pi\}$ and has a singularity at π. Observe that

$$\lim_{z\to\pi} (z-\pi)^2 f(z) = \lim_{z\to\pi} \frac{\sin z}{z-\pi} = \lim_{w\to 0} \frac{\sin(w+\pi)}{w},$$

where $w = z - \pi$. Since $\sin(w+\pi) = \sin w \cos \pi + \sin \pi \cos w = -\sin w$, we see that

$$\lim_{z\to\pi} (z-\pi)^2 f(z) = \lim_{w\to 0}\left(-\frac{\sin w}{w}\right) = -1,$$

by HB B4 1.4, p55. This limit exists and is non-zero, so f has a pole of order two at π, by HB B4 3.2, p58. 4

(c) The function

$$f(z) = e^{1/(z-i)}$$

is analytic on $\mathbb{C} - \{i\}$ and has a singularity at i. We know that

$$e^w = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots, \quad \text{for } w \in \mathbb{C}.$$

Substituting $w = 1/(z-i)$ gives

$$e^{1/(z-i)} = 1 + \frac{1}{z-i} + \frac{1}{2!(z-i)^2} + \frac{1}{3!(z-i)^3} + \cdots, \quad \text{for } z \neq i.$$

This is the Laurent series about i for f. It has infinitely many non-zero coefficients in its singular part, so f has an essential singularity at i, by HB B4 2.10(c), p57. 3

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Question 3

The function f is given by a geometric series in $z/3$. If $|z| < 3$, then $|z/3| < 1$, so

$$f(z) = \sum_{n=0}^{\infty} \left(\frac{z}{3}\right)^n = \frac{1}{1 - z/3} = \frac{3}{3 - z}.$$

The function g is given by a geometric series in $3/z$. If $|z| > 3$, then $|3/z| < 1$, so

$$g(z) = -\frac{3}{z} \sum_{n=0}^{\infty} \left(\frac{3}{z}\right)^n = -\frac{3}{z} \times \frac{1}{1 - 3/z} = \frac{3}{3 - z}.$$

Let h be the analytic function

$$h(z) = \frac{3}{3 - z} \quad (z \in \mathbb{C} - \{3\}).$$

Define $\mathcal{R} = \{z : |z| < 3\}$ and $\mathcal{S} = \{z : |z| > 3\}$. Both these regions are subsets of $\mathbb{C} - \{3\}$, so

$$f(z) = h(z), \quad \text{for } z \in \mathcal{R} \cap (\mathbb{C} - \{3\})$$

and

$$g(z) = h(z), \quad \text{for } z \in \mathcal{S} \cap (\mathbb{C} - \{3\}).$$

It follows that $(f, \mathcal{R})$, $(h, \mathbb{C} - \{3\})$, $(g, \mathcal{S})$ is a chain of functions. The functions $(f, \mathcal{R})$ and $(g, \mathcal{S})$ are not direct analytic continuations of each other because $\mathcal{R} \cap \mathcal{S} = \varnothing$. Hence these two functions are indirect analytic continuations of each other. 10

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Question 4

(a) Let $f(z) = 2z^3 + 3z^2 - 12z$.

(i) The function f is analytic on $\mathbb{C}$, and the Taylor series about $\alpha = 0$ for f is

$$f(z) = -12z + 3z^2 + 2z^3.$$

Since the coefficient of z is non-zero we see from the Local Mapping Theorem that f is one-to-one near 0. 2

(ii) Observe that

$$f'(z) = 6z^2 + 6z - 12, \quad \text{so} \quad f'(1) = 6 + 6 - 12 = 0,$$

$$f''(z) = 12z + 6, \quad \text{so} \quad f''(1) = 12 + 6 = 18 \neq 0.$$

Since f is analytic on $\mathbb{C}$ we see from HB C2 3.5, p68, that f is two-to-one near 1. 3

(b) Let $\phi_n(z) = e^z/n^2$, for $n = 1, 2, \ldots$, and let $E = \{z : 0 \leq \operatorname{Re} z \leq 1\}$. If $z = x + iy \in E$, then $0 \leq x \leq 1$, so

$$|\phi_n(z)| = \left|\frac{e^z}{n^2}\right| = \frac{e^x}{n^2} \leq \frac{e}{n^2}, \quad \text{for } n = 1, 2, \ldots.$$

Since

$$\sum_{n=1}^{\infty} \frac{e}{n^2} = e \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges, by HB B3 1.9, p47, we see that

$$\sum_{n=1}^{\infty} \phi_n(z) = \sum_{n=1}^{\infty} \frac{e^z}{n^2}$$

is uniformly convergent on E, by the M-test. 5

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Question 5

(a) The conjugate velocity function is

$$\overline{q}(z) = \frac{(z-i)^2}{z}.$$

Let Γ be the unit circle $\{z : |z| = 1\}$. The Circulation and Flux Contour Integral tells us that

$$\mathcal{C}_\Gamma + i\mathcal{F}_\Gamma = \int_\Gamma \frac{(z-i)^2}{z}\, dz.$$

We can evaluate this integral using the Residue Theorem. By the Cover-up Rule,

$$\operatorname{Res}(\overline{q}, 0) = (-i)^2 = -1.$$

Hence

$$\mathcal{C}_\Gamma + i\mathcal{F}_\Gamma = 2\pi i \times (-1) = -2\pi i.$$

Therefore $\mathcal{C}_\Gamma = 0$, so 0 is not a vortex, and $\mathcal{F}_\Gamma = -2\pi < 0$, so 0 is a sink. 5

(b) Let $\mathcal{R} = \{z : 0 < |z| < 1\}$ and $\mathcal{S} = \{z : |z| > 1\}$. By HB D1 3.2(b), p85, the Joukowski function J is a one-to-one conformal mapping from $\mathcal{S}$ onto $\mathbb{C} - [-2, 2]$. So we seek a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathcal{S}$, which we will compose with J. The Möbius transformation

$$f(z) = \frac{1}{z}$$

is such a function, because Möbius transformations are one-to-one conformal mappings on $\widehat{\mathbb{C}}$, and

$$0 < |z| < 1 \iff |f(z)| > 1 \text{ and } f(z) \neq \infty,$$

so $z \in \mathcal{R}$ if and only if $f(z) \in \mathcal{S}$.

It follows that the composite mapping

$$g(z) = J(f(z)) = 1/z + \frac{1}{1/z} = \frac{1}{z} + z$$

is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathbb{C} - [-2, 2]$. That is, J itself is a one-to-one conformal mapping from $\mathcal{R}$ onto $\mathbb{C} - [-2, 2]$. 5

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Question 6

(a) Observe that

$$\begin{aligned}
&f(e^{i\pi/11}) = e^{3i\pi/11}\\
&f(e^{3i\pi/11}) = e^{9i\pi/11}\\
&f(e^{9i\pi/11}) = e^{27i\pi/11} = e^{5i\pi/11}\\
&f(e^{5i\pi/11}) = e^{15i\pi/11}\\
&f(e^{15i\pi/11}) = e^{45i\pi/11} = e^{i\pi/11}.
\end{aligned}$$

Hence $e^{i\pi/11}$ is a periodic point of f, with period 5.

Observe that $f'(z) = 3z^2$. By HB D2 2.11(a), p90, the multiplier of the 5-cycle is

$$\begin{aligned}
&(f^5)'(e^{i\pi/11})\\
&= f'(e^{i\pi/11}) \times f'(e^{3i\pi/11}) \times f'(e^{9i\pi/11}) \times f'(e^{5i\pi/11}) \times f'(e^{15i\pi/11})\\
&= 3^5(e^{i\pi/11} \times e^{3i\pi/11} \times e^{9i\pi/11} \times e^{5i\pi/11} \times e^{15i\pi/11})^2\\
&= 3^5 e^{2i\pi(1+3+9+5+15)/11} = 3^5.
\end{aligned}$$

Since $|(f^5)'(e^{i\pi/11})| > 1$, the 5-cycle is repelling. 6

(b) Observe that $c = -2$ belongs to M, by HB D2 4.7(c), p92. We will prove that $ic = -2i \notin M$. We have

$$\begin{aligned}
&P_c(0) = -2i,\\
&P_c^2(0) = (-2i)^2 - 2i = -4 - 2i.
\end{aligned}$$

Hence

$$|P_c^2(0)| = |-4 - 2i| = \sqrt{4^2 + 2^2} > 2.$$

It follows from HB D2 4.6, p92, that $-2i \notin M$. 4

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Question 7

(a) • This implication is true.

If A and B are compact, then they are closed and bounded. By the Combination Rules for Closed Sets, the set $A \cap B$ is closed, because A and B are closed. Also, $A \cap B$ is bounded, since it is contained in A. Therefore $A \cap B$ is compact.

• This implication is false.

For example, $A = \{z : -1 < \operatorname{Re} z < 1\}$ and $B = \{z : |z| > 2\}$ are both regions, but $A \cap B = \{z : -1 < \operatorname{Re} z < 1, |z| > 2\}$ is not a region because it is not path connected, since there is no path lying in $A \cap B$ from the point $3i$ to the point $-3i$.

• This implication is true.

If A and B are compact, then they are closed and bounded. By the Combination Rules for Closed Sets, the set $A \cup B$ is closed, because A and B are closed. Next, since A and B are bounded, they are each contained in a closed disc. If we choose a closed disc D that contains both of these closed discs, then D contains $A \cup B$ also. Hence $A \cup B$ is bounded.

Therefore $A \cup B$ is compact.

• This implication is false.

For example, $A = \{z : \operatorname{Re} z < 1\}$ and $B = \{z : \operatorname{Re} z > 1\}$ are both regions, but $A \cup B = \{z : |\operatorname{Re} z| > 1\}$ is not a region because it is not path connected, since there is no path lying in $A \cup B$ from the point 2 to the point -2. 10

(b) (i) Let $z = x + iy$. Since $z(1 + \overline{z}) = z + |z|^2$, we have

$$f(z) = z + |z|^2 = x + iy + x^2 + y^2 = (x + x^2 + y^2) + iy.$$

Define

$$u(x, y) = x + x^2 + y^2 \quad \text{and} \quad v(x, y) = y.$$

Then $f(z) = u(x, y) + iv(x, y)$, and

$$\frac{\partial u}{\partial x}(x, y) = 1 + 2x,$$

$$\frac{\partial u}{\partial y}(x, y) = 2y,$$

$$\frac{\partial v}{\partial x}(x, y) = 0,$$

$$\frac{\partial v}{\partial y}(x, y) = 1.$$

The first Cauchy–Riemann equation is

$$\frac{\partial u}{\partial x}(x, y) = \frac{\partial v}{\partial y}(x, y) \iff 1 + 2x = 1 \iff x = 0.$$

The second Cauchy–Riemann equation is

$$\frac{\partial u}{\partial y}(x, y) = -\frac{\partial v}{\partial x}(x, y) \iff 2y = 0 \iff y = 0.$$

Hence both the Cauchy–Riemann equations are satisfied if and only if $x = y = 0$.

Since the partial derivatives exist and are continuous on $\mathbb{C}$, and the Cauchy–Riemann equations are satisfied at $z = 0$, we see from the Cauchy–Riemann Converse Theorem that f is differentiable at 0.

However, the Cauchy–Riemann equations fail at points other than 0, so f is not differentiable at any point other than 0, by the Cauchy–Riemann Theorem. It follows that f is not differentiable on a region containing 0, so it is not analytic at 0. 8

(ii) By the Cauchy–Riemann Converse Theorem,

$$f'(0) = \frac{\partial u}{\partial x}(0,0) + i\frac{\partial v}{\partial x}(0,0) = 1 + i0 = 1.$$ 2

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Question 8

(a) (i) By the Radius of Convergence Formula, the radius of convergence is

$$\begin{aligned} R &= \lim_{n\to\infty}\left|\frac{e^{in}/n!}{e^{i(n+1)}/(n+1)!}\right| \\ &= \lim_{n\to\infty}\left|\frac{n+1}{e^{i}}\right| \\ &= \lim_{n\to\infty}(n+1) = \infty. \end{aligned}$$ 3

(ii) By the Radius of Convergence Formula, the radius of convergence is

$$\begin{aligned} R &= \lim_{n\to\infty}\left|\frac{3^n+\cos n}{3^{n+1}+\cos(n+1)}\right| \\ &= \lim_{n\to\infty}\left|\frac{1+(\cos n)/3^n}{3+(\cos(n+1))/3^n}\right|. \end{aligned}$$

Now $|\cos n| \le 1$, for $n = 1, 2, \ldots$, hence

$$(\cos n)/3^n \to 0 \quad \text{and} \quad (\cos(n+1))/3^n \to 0 \quad \text{as } n \to \infty.$$

Therefore $R = 1/3$. 3

(b) (i) We have

$$\cos w = 1 - \frac{1}{2}w^2 + \frac{1}{24}w^4 - \cdots, \quad \text{for } w \in \mathbb{C},$$

$$\exp z = 1 + z + \frac{1}{2}z^2 + \cdots, \quad \text{for } z \in \mathbb{C}.$$

Hence

$$z\exp z = z + z^2 + \frac{1}{2}z^3 + \cdots, \quad \text{for } z \in \mathbb{C}.$$

Let $w = z\exp z$. Since $0\exp 0 = 0$, we can apply the Composition Rule for Power Series to give

$$\begin{aligned} \cos(z\exp z) &= 1 - \frac{1}{2}\left(z + z^2 + \frac{1}{2}z^3 + \cdots\right)^2 + \frac{1}{24}(z+\cdots)^4 - \cdots \\ &= 1 - \frac{1}{2}\left(z^2 + 2z^3 + 2z^4 + \cdots\right) + \frac{1}{24}\left(z^4 + \cdots\right) - \cdots \\ &= 1 - \frac{1}{2}z^2 - z^3 - \frac{23}{24}z^4 - \cdots. \end{aligned}$$

Since g is an entire function, this Taylor series converges to $g(z)$ for each $z \in \mathbb{C}$, by HB B3 3.5, p51. 5

(ii) The function $f(z) = z^3 g(1/z)$ is analytic on the simply connected region $\mathbb{C}$ except for a singularity at 0. By part (b)(i) we have

$$\begin{aligned} z^3 g(1/z) &= z^3\left(1 - \frac{1}{2z^2} - \frac{1}{z^3} - \frac{23}{24z^4} - \cdots\right) \\ &= z^3 - \frac{z}{2} - 1 - \frac{23}{24z} - \cdots, \end{aligned}$$

for $z \in \mathbb{C} - \{0\}$. Hence

$$\operatorname{Res}(f, 0) = -\frac{23}{24}.$$

Applying the Residue Theorem we see that

$$\int_C z^3 g(1/z)\,dz = 2\pi i \times \left(-\frac{23}{24}\right) = -\frac{23\pi i}{12}.$$ 4

(c) We are given that f is bounded on the strip $S = \{z : 0 \le \operatorname{Re} z \le 1\}$, so there is a positive constant K such that

$$|f(z)| \le K, \quad \text{for } 0 \le \operatorname{Re} z \le 1.$$

Now consider any complex number $z = x + iy$. Then $n \le x < n+1$, for some integer n, so $0 \le x - n < 1$. Since $\operatorname{Re}(z-n) = x - n$, we see that $z - n \in S$. We are told that $f(z-n) = f(z)$, so

$$|f(z)| = |f(z-n)| \le K.$$

It follows that f is bounded on $\mathbb{C}$, and it is entire, so it must be a constant function, by Liouville's Theorem. 5

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Question 9

(a) By HB C1 4.7, p63, the Laurent series about 0 for cosec is

$$\operatorname{cosec} z = \frac{1}{z} + \frac{1}{6}z + \cdots,$$

so

$$\operatorname{cosec} \pi z = \frac{1}{\pi z} + \frac{1}{6}\pi z + \cdots.$$

Also, for $|9z^2| < 1$, we have the binomial series

$$(9z^2+1)^{-1} = 1 - (9z^2) + \cdots = 1 - 9z^2 + \cdots.$$

Hence the Laurent series about 0 for f is

$$\begin{aligned} f(z) &= \frac{\pi \operatorname{cosec} \pi z}{9z^2+1} \\ &= \pi\left(\frac{1}{\pi z} + \frac{1}{6}\pi z + \cdots\right)(1 - 9z^2 + \cdots) \\ &= \pi\left(\frac{1}{\pi z} + \left(\frac{\pi}{6} - \frac{9}{\pi}\right)z + \cdots\right) \\ &= \frac{1}{z} + \left(\frac{\pi^2}{6} - 9\right)z + \cdots. \end{aligned}$$ 4

(b) From the Laurent series found in part (a) we can see that

$$\operatorname{Res}(f, 0) = 1.$$

For the residue at $\frac{1}{3}i$ we first write f as

$$f(z) = \frac{\pi \operatorname{cosec} \pi z}{9\left(z - \frac{1}{3}i\right)\left(z + \frac{1}{3}i\right)},$$

and then apply the Cover-up Rule to give

$$\operatorname{Res}\left(f, \tfrac{1}{3}i\right) = \frac{\pi \operatorname{cosec}(\pi i/3)}{9 \times \frac{2}{3}i} = \frac{\pi}{6i \times i \sinh \pi/3} = -\frac{\pi}{6 \sinh \pi/3}.$$

Since f is an odd function, we see from HB C1 1.1, p59, that

$$\operatorname{Res}\left(f, -\tfrac{1}{3}i\right) = \operatorname{Res}\left(f, \tfrac{1}{3}i\right) = -\frac{\pi}{6 \sinh \pi/3}.$$ 5

(c) The function

$$h(z) = \frac{1}{9z^2+1}$$

is an even function that is analytic on $\mathbb{C}$ except for poles at $\pm\frac{1}{3}i$, neither of which is an integer. Let S_N be the square contour with vertices at $\left(N + \frac{1}{2}\right)(\pm 1 \pm i)$. If $z \in S_N$, then $|z| > N$. Hence, for $z \in S_N$, we see from the backwards form of the Triangle Inequality that

$$|9z^2 + 1| \geq |9z^2| - 1 \geq 9N^2 - 1.$$

By HB C1 4.6, p63, we know that $|\operatorname{cosec} \pi z| \leq 1$, for $z \in S_N$. Therefore

$$|f(z)| = \left|\frac{\pi \operatorname{cosec} \pi z}{9z^2+1}\right| \leq \frac{\pi}{9N^2 - 1}, \quad \text{for } z \in S_N.$$

We can now apply the Estimation Theorem, using the fact that S_N has length $4(2N+1)$, to obtain

$$\left|\int_{S_N} f(z)\,dz\right| \le \frac{\pi}{9N^2-1} \times 4(2N+1) = \frac{4\pi(2N+1)}{9N^2-1}.$$

Now

$$\frac{4\pi(2N+1)}{9N^2-1} = \frac{4\pi(2/N+1/N^2)}{9-1/N^2} \to 0 \text{ as } N \to \infty,$$

so

$$\lim_{N\to\infty} \int_{S_N} f(z)\,dz = 0.$$

We can now apply HB C1 4.4, p63, to deduce that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^n}{9n^2+1} &= -\frac{1}{2}\left(\operatorname{Res}(f,0) + \operatorname{Res}\left(f, \tfrac{1}{3}i\right) + \operatorname{Res}\left(f, -\tfrac{1}{3}i\right)\right) \\ &= -\frac{1}{2}\left(1 - \frac{\pi}{3\sinh \pi/3}\right) \\ &= -\frac{1}{2} + \frac{\pi}{6\sinh \pi/3}.\end{aligned}$$ 8

(d) Since $n \longmapsto (-1)^n/(9n^2+1)$ is an even function, we see from part (c) that

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{9n^2+1} &= \frac{(-1)^0}{9\times 0^2+1} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{9n^2+1} \\ &= 1 + 2\left(-\frac{1}{2} + \frac{\pi}{6\sinh \pi/3}\right) \\ &= \frac{\pi}{3\sinh \pi/3}.\end{aligned}$$ 3

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